Classfying Handwritten Digits Through K-Means Clustering

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*Abstract*—In this case study we used the skills and methods we have learned in linear algebra and MATLAB to simulate dynamical systems and explore strategies to control them. This case study links topics of linear dynamical systems and control engineering and applies them to model the real-world issue of COVID-19 pandemic spread dynamics.

# Introduction (*Heading 1*)

This project utilizes concepts of linear dynamical systems to approach modelling the issue of pandemic spread, using the recent outbreak of the novel COVID-19 virus as data. As this project utilizes a myriad of linear algebraic techniques and topics learned in ESE 105, it enforces the learning done in class and allows a practical application of the subjects explored. Particularly, we seek to implement a classic pandemic spread model called the SIRD model. We then aim to fit it to local COVID-19 spread data. After such, we generalize the SIRD model to account for the individuals in lockdown, account for vaccinations, and to create a larger model which can highlight the number of people in each local region, and how people can travel between regions.

# Methods

As aforementioned in the introduction, our overall task comes fourfold. Firstly, we aim to implement a classic model describing pandemic spread, named the SIRD model. Each state in this model is represented by a four-dimensional vector, each element in which represents the fractional number of people in a population that are susceptible, infected, recovered, and deceased from the disease. After such implementation, we aim to fit this model to data from the ongoing COVID-19 pandemic. We aim to achieve this particularly within the context of our local regions: St. Louis, Jefferson City, and Springfield, MO. From such application, we can make inferences surrounding real-world COVID-19 dynamics based on our model. Thirdly, we seek to generalize the SIRD model to create a new, SLIRD model. This model instead has 5 states, with the addition of the ‘L’ state highlighting the proportion of the population in lockdown, with the implication that the population belonging to the other states (SIR) are currently not locked down. Finally, we seek to create an intervention strategy. We may choose to implement lockdown laws, vaccination deployments, or treatment options. We then seek to optimize this solution to better the modelled pandemic situation.

## SIRD Modelling

Firstly, we must create a SIRD model to represent current pandemic dynamics. As highlighted in Section 9.3 of Boyd and Vendenberghe’s Introduction to Applied Linear Algebra, the SIRD model takes the form of a linear dynamical system. Such a system possesses a state vector representing the state of the pandemic at each time step. To model the next time step, the transition matrix is used, which represents the development of the pandemic at any timestep .

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Concretely, the state vector at time is defined , each element of which corresponds to a fractional quantity of people susceptible, infected, recovered (and hence immune), and dead. The transition vector , after the multiplication , yields another vector .

The transition matrix can be modelled as follows. We have defined a few constants which can help calculate the transition matrix. Particularly, the constants , , and represent the rates of infection, fatality, and recovery, respectively. Hence, the transition matrix is defined as follows:

Each element in represents one transition between any element in SIRD. The columns and rows may be indexed with S, I, R, D for rows / columns 1, 2, 3, and 4. For example the element at column 2, row 3 () represents the rate at which I transfers to R.

Following (1), it is evident that any progression in timestep can be modelled as a simple matrix multiplication between a state vector and a transition matrix. Hence, given any arbitrary state at any time step, as well as a transition matrix, the state space of any time after the initial one given can be calculated. Namely,

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We modelled our SIRD based on the concept given by (2). Using this model, an example epidemic evolution trend can be seen in Fig. 1.

Chart

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Fig. 1. Evolution trend of a sample epidemic based on the linear SIRD model

Despite changing the coefficients in the transition matrix, the model tends to converge with the majority of the population recovered and a small fraction deceased, as seen in Fig. 1. Of course, given an arbitrarily high death rate, it is also possible for a disease to kill faster than it is infecting, with the model hence converging with some fraction of population still susceptible. However, since this requires an unrealistically high lethality rate, we did not explore this approach. Instead, we concluded that in the supermajority of cases, the model converges with only recovered and deceased.

## COVID-19 Data Regression

Since modelling the SIRD model, we were able to fetch real COVID-19 data from years 2020 to 2021 in the local regions of St. Louis, Jefferson City, and Springfield, MO. We seek to regress our SIRD model to such real world data, and obtain transition matrices which model the spread of COVID-19.

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Fig. 2. COVID-19 Data in 3 regions since 2020-3-17

Fig. 2 highlights the gathered COVID-19 trends according to which we will fit our data. However, evident from the graph, there are a few different stages within the data which have different infection rates. Simply through visual estimation, the interval may represent three distinct stages of disease evolution. Hence, it will be more accurate in terms of our model to separate these segments and fit the model to them individually.

|  |  |  |  |
| --- | --- | --- | --- |
| **Time** | **[1,200]** | **[201,400]** | **[401,600]** |
| **Inf. Rate** | 0.0001 | 0.001 | 0.0003 |

Table 1. St. Louis infection rates at different time intervals

However, before fitting, we must first define a cost function which represents the difference between the gathered data and the modelled data. Matlab’s function can then minimize the cost function through tweaking coefficients in the transition matrix.

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In (3), , , represent the real deaths, infected, and recovered cases, while , , represent the deaths, infected, and recovered cases predicted by the model.

However, note that we were not actually given the recovery rate from the COVID-19 data. Instead, we have chosen to derive it as follows: The CDC highlights that on average, COVID-19 takes around a week to recover from. We have chosen to simplify this assumption: that the disease takes exactly one week to either kill or recover from. Therefore, the recovery rate at any time frame can simply be represented as the infected rate 7 days before subtracted by the current death rate. In other words,

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Fig. 3. Real and predicted infection, death, and recovery rates

Fig. 3 models the interval . As evident from the graph, the general fit between the predicted and measured graphs are quite close, suggesting that the cost function can accurately model the residual between predicted and real data.

There is a clear difference between the infection rates in the intervals . Particularly, for example, the St. Louis infection rates are as below:

## Lockdown and Vaccination Generalization

In reality, the COVID-19 situation could be represented as more than just the states in SIRD. Namely, vaccination and lockdown are two important rates which can function to greatly affect the epidemic dynamics. To accommodate for this, we change our model by adding one other state L, into the SLIRD model. We have hence added the transfer coefficients and, to represent the rate at which individuals exit and enter lockdown. Furthermore, we add another transition coefficient , which models the rate at which susceptible individuals can be immunized, and hence transferred to the recovered group. With this, the transition matrix is now .

Note that this is under the assumption that both susceptible and infected individuals enter lockdown at the same rate, and that only susceptible individuals exit (as in present policy, people who test positive in COVID-19 cannot exit quarantine).

Furthermore, after fitting the model again with vaccination in mind, we could compare an estimate of vaccination trends between the cities. As seen in Table 2, Jefferson City seems to have the highest overall vaccination rate by far.

|  |  |  |  |
| --- | --- | --- | --- |
| **Time** | **Jefferson City** | **St. Louis** | **Springfield** |
| **Vac. Rate** | 0.0062 | 0.0012 | 0.0011 |

Table 2. Vaccination rates across cities

We were unfortunately unable to fully complete the integrated model in which the three different regions could be modelled together. Our methods were unsuccessful at finding realistic rates of population transfer rates in between the regions. Despite this, we are still able to glean valuable insight into present COVID-19 pandemic trends through investigating the individual region trends as seen in Fig. 2.

## Model Interpretation

From investigating Fig. 2, a very clear correlation between the regions can be established. In particularly, the trends between the regions were nearly identical in their rise and fall in infection rates. Particularly, through calculating the correlation coefficients between the infection rates of the three regions, the following table is obtained:

|  |  |  |  |
| --- | --- | --- | --- |
| **City Name** | **Jefferson City** | **St. Louis** | **Springfield** |
| **Jefferson City** | 1 | 0.9864 | 0.9957 |
| **St. Louis** | 0.9864 | 1 | 0.9818 |
| **Springfield** | 0.9957 | 0.9818 | 1 |

Table 3. Correlation Coefficients between infection rates in cities

Evident through Table 3’s correlation coefficients between the cities’ infection rates, there is an extremely strong positive relation between the infection rates in the three regions. This suggests that there was either very frequent travel in between the regions, or that there was some external factor which have equally impacted the regions at the same time. This is not an unexpected phenomenon, since not only are these regions geographically close and connected, but also because the historical events which drove COVID-19 trends in the last year, such as the delta variant hitting the US and the implementation of stricter lockdown regulations, were applied to the entirety of Missouri. For example, the large rise in infection cases marked by occurs at the end of 2020 matches quite closely with the impact of the delta variant to the U.S. This large rise in infection rates is followed by a large fall, which matches stricter mask and quarantine regulations following the delta variant.

# Policy discussion

As described in the methodology, we have been tasked with coming up with a policy in order to minimize COVID-19 cases. Concretely, such a policy corresponds to varying the coefficients in the transition matrix , with each changed coefficient corresponding to differing policies in real life. For example, a decreased infection rate in may signify a mask-wearing or social distancing policy, whereas a decrease in death rate may correspond to increased emergency care for those already infected.

As shown in Fig. 4, when evaluated against the test set of 200 images, the model with performed the best, with an accuracy of 153 predictions. In choosing our value of choice, we have also added a penalty for each additional we used. When not considering the outlier classification, according to the scoring metric, score is awarded as according to (3):

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Fig. 4. Accurate predictions in test set performed by each model of varying K values

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Hence we have modified our program to use the model which yields the greatest score, by subtracting half of the from each model’s accuracy to yield each program’s score.

In this case, the optimal model is still at .

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Fig. 5. Score yielded by each model

Now, we plot the K-Means cost function, as shown in (1), as a function of iterations. As expected the classic shape of the steep drop into a shallowing decline is seen in Fig. 6.

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Fig. 7. Predicted labels and true labels performed by

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Fig. 6. K-Means cost of the model as a function of iterations

To visualize the performance of our model, we have shown a representation of the labels as predicted by our model, and the correct labels in the test set in Fig. 7.

However, to evaluate the correctness of our model, we first had to assign each centroid to a digit which it represented. We did this by checking each centroid’s associated data points during training. Since the training set is labelled with the correct digits, we have taken each centroid’s digit representation to equal the mode of the data points’ digit representations. In other words, each centroid represents the digit that occurs most commonly in its associated data points. With this, we have also displayed the centroids of our model in Fig. 8.

Calendar

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Fig. 8. Centroids and digit representations of the model

With these centroids, we were able to correctly evaluate the digits of approximately 75% of the data points we were presented, at least in the test set.

# Conclusion

From our experimentation, we have found a set of centroids which can correctly classify approximately 75% of an MNIST hand-written digits dataset. We have also derived a method to identify potential outliers based on the k-means architecture.

Despite our work, however, our k-means clustering algorithm does have its drawbacks – some inherent within the algorithm, and others in the way we have structured our model. Firstly, k-means clustering is very computationally expensive. There is no guarantee that the algorithm will converge in any given amount of time, with a very high time complexity of , where represents the number of data points in the input data set [1]. This time complexity is further compounded by our choice to evaluate the k-means algorithm for 10 values of , which does yield a better model, but takes a lot more time. Secondly, a few assumptions were made in our program, the main assumption being that each digit could be identified according to its cluster. For example, the differences between how some “1”s and “7”s are written are very miniscule, and the two digits could very much be clustered together with indistinguishable differences, based on their similarity in that they both have a vertical line in the center. Finally, the position of the handwritten digits also has a significant effect on the classification. A digit written that is offset by a few pixels can have significant implications on the classification, as the k-means algorithm takes each pixel as an independent dimension, and will hence read two offset images of the same digit to have little to no similarity. To seek improvement, we could use a smaller number of trials in to narrow down the optimal value, which trades optimality against time complexity. To solve the offsetting problem, we may perform some data manipulation to center each digit in the center.

##### References

1. M. K. Pakhira, "A Linear Time-Complexity k-Means Algorithm Using Cluster Shifting," 2014 International Conference on Computational Intelligence and Communication Networks, 2014, pp. 1047-1051, doi: 10.1109/CICN.2014.220.